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Students' Growth of Mathematical Understanding in Solving Derivative Problem

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Abstract: *Qualitative research method were employed to explore growth of mathematical understanding of two first year students of mathematics education in solving derivative problem based on growth of mathematical understanding model by Pirie-Kieren. The results showed that subjects' growth of mathematical understanding spanned layers of primitive knowing, image making, image having, property noticing, formalizing, and observing. Despite of the same spanned layers, subjects' had different path of understanding growth. Both subjects couldn't fold back by themselves, yet they needed intervention.*
Keywords: *The growth of understanding, Pirie-Kieren, Derivative.*

I. Introduction

The definition of mathematical understanding evolves time to time (Meel, 2003). Constructivism approach defines mathematical understanding as a continue process of organizing one knowledge structure (von Glasersfeld, 1989). One of model that correspond with constructivism approach is Pirie-Kieren growth of mathematical understanding model. The model is considered to be able to describe and analyze growth of mathematical understanding effectively (Kyriakides, 2010; Manu, 2005; Martin & Pirie, 2003; Walter & Gibbon, 2010). According to Pirie-Kieren(1991), growth of mathematical understanding is characterized as leveled but nonlinear, dynamic, and transcendentally recursive process or organizing one knowledge structure.

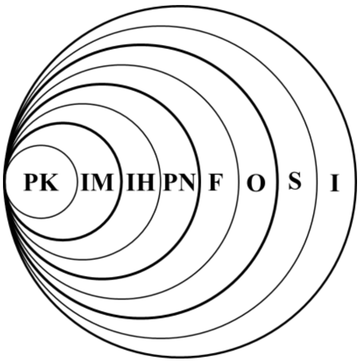


Figure 1. Representation of growth of mathematical understanding model by Pirie-Kieren

Leveled characteristic of the model is showed by eight layers of understanding, namely *primitive knowing* (PK), *image making* (IM), *image having* (IH), *property noticing* (PN), *formalizing* (F), *observing* (O), and *inventizing* (I), as presented in Figure 1. Primitive knowing is the core layer. It consists of one's former knowledges that's used to understand a new concept or solve a new problem. When one try to make images, physical or mental, to understand the problem, they are in the second layer, that is image making. Ones can also adjust the image they had to a new situation in the problem given (Kieren, Pirie, & Calvert, 2002). After making and comparing the images, one state the most suitable image to understand the problem. In the next layer, image having, one can formulate mental that can be used to solve the problem. Property noticing is the next layer where they check, compare, and establish connection between ideas or concepts according to the emerging properties. In formalizing layer, one can abstract a method or formula that can be applied in general. In the sixth layer, observing, one are able to coordinate mathematical theorems and concept to solve the problem. In the next layer, structuring, they can determine relation between theorems and prove them using formal argumentation. Finally, in the outer layer, inventising, one already have a complete understanding structure about the concept and are able to pose questions that lead to a 'new concept'.

The layers are not linear, that is the inner layer doesn't imply low mathematical understanding or skill and vice versa. Moreover, growth of mathematical understanding are not always in one direction, from inner to outer layer. When one get deadlock in solving problem, they may come back to their inner layer in order to get better understanding. In Pirie-Kieren model, this phenomenon is called folding back. Folding back indicates the dynamic and transcendentally recursive process of growth of mathematical understanding.

Another feature of Pirie-Kieren model is intervention. Intervention is either internal or external action to stimulate one's understanding. Pirie-Kieren classify intervention into provocative, invocative, and validative. Provocative intervention encourages one to consider an outer layer from their current layer of understanding. Invocative intervention prompts one to notice that they need to do folding back or go to inner layer. While validative intervention makes them express their understanding.

Students of mathematics education are demanded to have good understanding about mathematical concepts. One of them is derivative. An observation that was conducted earlier showed that students of mathematics education in State University of Malang had difficulty in solving derivative problems, especially problems that involved fundamental concept of derivative. The main cause was students' lack of conceptual understanding. Infrequent exploration to non-routine problem contributed to this matter whereas non-routine problem scientifically approved to stimulate growth of understanding.

The research tried to explore growth of mathematical understanding of mathematics education students when they solved non-routine derivative problem. The research is expected to contribute to study of growth of understanding in derivative.

II. Method

The research was qualitative study. The subjects were two first year of mathematics education students in State University of Malang that took calculus. Both subjects were asked to solve a problem as follows. Let$f(x) = \begin{cases} mx+b, x<2 \\ x^2 \quad\quad, x \geq 2 \end{cases}$. Determine the value of m and b so that f is differentiable in its domain (Varberg, Purcell, & Rigdonn, 2000). Even though the problem was in their text book, they hadn't solve this problem. In fact, survey said that they felt unfamiliar with that kind of problem. Thus, the problem was non routine.

Subjects were requested to solve the problem in writing and think aloud. When solving it, subjects were allowed to open their notebook or text book. Everything was recorded. Semi structured interview was also conducted to get better sight to students' understanding. So, various collected data were students' written test result, video, and interview transcripts. By means of the data, students' behavior was mapped into layers of understanding and path of understanding growth was described based on Pirie-Kieren model.

III. Results

First Subject

After reading the problem, first subject decided that he needed to find conditions for differentiable function. He said that in order to be differentiable, function had to be continue, showing that he already had an image in his head. Thus, first subject was in layer of image having. Being asked for the reason, first subject could say nothing. Then, he tried to recollect any information he had in his notebook and textbook for about five minutes to give correct argumentation. This implied first subject folded back to image making. Yet, his collecting wasn't fruitful. After seven minutes of ineffective collecting, researcher asked him to consider formal definition of derivative. Researcher gave invocative intervention so that first subject folded back to primitive knowing.

Subject read the definition of derivative and explanation below it in his text book. It said that 'if the limit exists, the function is differentiable in c'. The explanation gave him a new image, that is in order to be differentiable everywhere, the limit had to be exists everywhere. Then again, subject was in image having layer. Without any further explanation, he calculated the value of $\lim_{x\to 2} f(x)$. He got correct calculation by noticing equality of left and right hand limit. His result was $2m + b = 4$ as shown in Figure 2a. Even though it hadn't been a complete result, he demonstrated application of definition to solve the problem, showing that subject was in layer of observing. Then, researcher asked for further explanation on the reason of such calculation. Subject explained that it was because of the information below the definition of derivative in the textbook, 'if the limit exists, the function is differentiable in c'. This showed that subject didn't have correct image since he had misconception about the function that its limit needed to be calculated based on the information given.

Researcher asked subject to reread the definition of derivative, interpret the meaning of the explanation, as well as comparing that to his work. Researcher did invocative intervention that lead the subject to layer of primitive knowing. Later, subject realized that he made mistake in understanding the information. Thus, he calculated the value of $f'(2)$ using definition as shown in Figure 2b. Subject was in observing again with more precise understanding.

Thereafter, subject got another deadlock. He stopped scratching his work for about five minutes. Researcher asked the reason why he used $f(x) = x^2$not $f(x) = mx + b$. The question gave him cue to realize the need to calculate $f'(2)$ with $f(x) = mx + b$ like he did in Figure 2c. Later, he explain that it should be done due to equality of right and left hand limit. He said that h approaching 0 from both left and right implied the necessity to consider the function from both left and right of $x = 2$. Subject's explanation as a result of invocative intervention indicate that he was in property noticing. Subject was aware that derivative was defined with notation of limit, so properties of limit was also possessed by derivative. As subject continued his work, he finally got the value of $m$and b, as shown in Figure 2d. Then again, he was in observing layer.

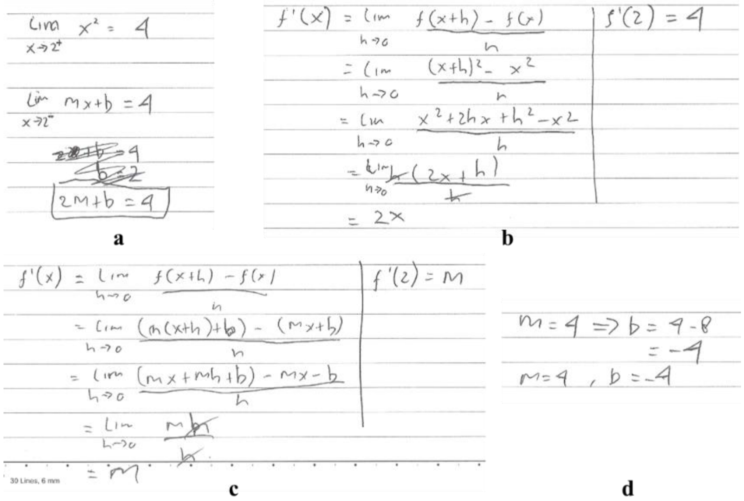


Figure 2. First Subject's Work

There was only one thing to be fixed, that the result in Figure 2a was gotten by misunderstanding. Researcher asked for subject confirmation. First subject stopped for long time before he stated again that to be differentiable, function needed to be continue. He opened his text book and found a theorem saying "if $f'(c)$ exists then f continue in c" (Varberg, Purcell, & Rigdonn, 2000). After reading the theorem, subject looked in doubt. He considered the premise and conclusion of the theorem and then stated that the theorem was relevant to the problem. Thus, he said that the answer he got was valid. The episode was inferred from interview transcript below.

Researcher (R) : *The theorem said that if f is differentiable in c then f is continue on c. Does that implies continuity is condition for a function to be differentiable?*

Subject I (S1) : *Hmmm… Wait a minute. Continuity doesn't imply differentiability, like this one* (pointing graph of absolute value function in the textbook). *If f continue, the function may not be differentiable. How is it? It means that continuity is not the condition, doesn't it?* (stop for about three minutes). *In this theorem, continuity is the consequence, right?* (stop for long time).

R : *So?*

S1 : *Since the problem asked that f is differentiable, then it's the assumption that f is differentiable. Thus, it's already given. Because f is differentiable, by the theorem, f must be continue. Nah, so, my answer is correct, right?*

The transcript showed us that subject folded back to primitive knowing, that is understanding of logics of implication. This resulted on formalizing that convinced him that his answer was valid because it was consistent with the theorem. First subject growth of understanding was mapped into understanding layers of Pirie-Kieren as shown in Figure 3.

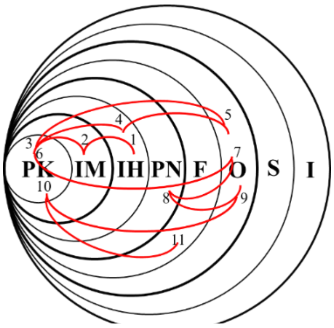


Figure 3. Path of First Subject's Growth of Understanding

Second Subject

Not much different with the first one, the second subject needed relatively long time to formulate steps to solve the problem. He opened his textbook. He tried to collect information that he could use. He was in image making layer. Second subject stared at theorem of differentiability implies continuity. He stated that since the function had to be differentiable everywhere, by the theorem, the function needed to be continue everywhere. Thus, subject already had a certain image to solve the problem, implying that he was in image having layer.

The researcher asked the subject about the continuity of function f in its domain. Conducted interview showed that subject hadn't mastered continuity concepts. The researcher did some intervention about it. Transcript of interview was shown as follow.

R : *What is a continue function?*

Subject II (S2) : *It's a function that don't snap.*

R : *What is meant by that?*

S2 : *I mean, the graph of function doesn't snap. There's no a jump in the graph.*

R : *Then, what about function f? Is it continue in its domain?*

S2 : (stopped for long time)

R : *Is it continue in* $x = 1$*?*

S2 : *Yes, it's in a line. So it's continue.*

R : *Is it continue in* $x = 5$*?*

S2 : *Mmmm, yes. It's in parabola, so it's continue.*

R : *Is there any point where f is not continue?*

S2 : (stopped for three minutes) *in* $x = 2$, *I guess. So, f needs to be continue there, right?*

R : *So?*

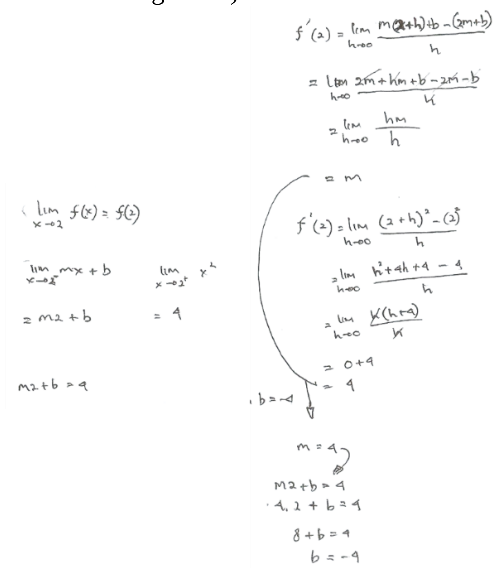
S2 : *Let me see the textbook again.* (opened page with definition of continue function) *Oh, I remember. The value must be the same from right and left, doesn't it?*

R : *What do you mean?*

S2 : *There's limit there. So, value of limit must be the same both from right and left hand.* (started to write $\lim_{x\to 2} f(x) = f(2)$ and stopped) *Wait a second, which part of function do I need?*

R : *Again, it's about what you mean by left and right earlier.*

S2 : *Ah, it must be the same when approaching 2 from left and right.* (stopped for a moment) *So, I need to use quadratic one when approaching from right and linier when approaching from left.* (continued his work to calculate $\lim_{x\to 2} f(x)$ as shown in Figure 4a)



a b

Figure 4. Second Subject's work

The vignette showed that second subject understood the continuity of function intuitively. This understanding wasn't enough to solve the problem. That led him to fold back to his inner layer of understanding. He did folding back to primitive knowing about continue function. Not simply collecting in his inner layer, he worked there. He made specific images to determine continuity of function f in some point in its domain. The folding back happened by researcher intervention. After making the images, he stated that to be continued everywhere, he needed to ensure the continuity of function in $x = 2$. Here, subject had formulated the procedure to solve the problem, thus, he was in layer of formalizing.

After formulating the procedure, subject folded back by himself to primitive knowing about formal definition of function. Noticing that definition of continue function involved limit reminded him that the concept of equity of left-hand and right-hand limit. He used these concepts to solve the problem. From primitive knowing, subject went to observing layer when he partially solve the problem.

Second subject got another deadlock and asked the researcher for next step. The researcher asked him to consider the formal definition of derivative. The researcher did invocative intervention so that subject folded back to primitive knowing. Subject red the definition in his textbook. Later, he identified the same properties of continuity and differentiability of function, namely using notation of limit. That implied subject was in property noticing layer. Using left-hand and right-hand limit, subject calculated the value of $f'(2)$. Finally, he got the value of m and b as shown in Figure 4b. Subject solved the problem completely, so he was already in observing layer again. Path of second subject's growth of understanding was mapped in Figure 5.

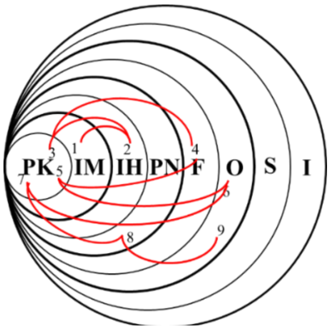


Figure 5. Path of Second Subject's Growth of Understanding

IV. Discussion

Both subjects' understanding spanned in the first six layers of understanding growth model of Pirie-Kieren. These layers were primitive knowing, image making, image having, property noticing, formalizing, and observing. None of them indicated to be in layer of structuring and inventising. Even though they have the same spanned layer of understanding, their path of understanding growth were different. The path of understanding growth was unique to the person, topic, and condition they were in. (Kieren, Pirie, & Calvert, 2002). For example, the first subject collected his understanding about the logic of implication statement, but the second one didn't. Meanwhile, the first subject didn't reconstruct the concept of continuity like the second subject did. This was caused by different former knowledge that they had.

Both subjects demonstrated folding back as one of feature in growth of understanding model of Pirie-Kieren. Through folding back to inner layer, subjects could go to outer layer or come back to his former layer with deeper understanding. It was important to notice that folding back wasn't equal to quick memory retrieval of fact or formula (Susiswo, 2014). As shown by first subject, he did activity of collecting in primitive knowing about formal definition of derivative and logics of implication statement. The collecting of information or former knowledge was vital to facilitate the growth of understanding (Pirie, Susan, & Lyndon, 2000). In addition to collecting, second subject indicated the activity of working in inner layer as another form of folding back (Martin, 2008) when he reconstructed concept of function continuity.

The most frequent folding back was going to primitive knowing layer. This layers included all initial knowledge or students'' past experiences that relevant with prerequisite concept to solve the problem. That implied the significance of understanding prerequisite concept in order to solve mathematical problem. This was reciprocal with a study conducted byBorgen& Manu(2002).

Another Pirie-Kieren growth of understanding feature emerged in observation was intervention. Most of folding back occurred by invocative intervention by the researcher. Subjects couldn't intervene themselves to fold back to inner layer. Studies by Martin (2008) and Sengul&Argat(2015) showed the same result. Intervention was an effective way to promote students' understanding when they had deadlock in solving problem (Borgen & Manu, 2002). Yet, the intervention should be done in the right time and condition otherwise it would block their understanding as notice by Towers (2002).

One factor contributed to growth of understanding was task selection. Problem solving was an appropriate way to facilitate connection between important concepts in derivative, for example continuity and limit. When students could connect these concept, they could develop their understanding and extend their skill in applying them effectively (NCTM, 2000). The structure of the problem could predetermine the spanned layer of understanding (Nillas, 2010). The result of research that subjects' understanding only spanned in six layers, was in consequence of the problem structure. Hence, careful selection of the problem was necessary to develop students' growth of understanding.

V. Conclusion

Subjects' growth of understanding spanned in layers of primitive knowing, image making, image having, property noticing, formalizing, and observing. Both subject demonstrated folding back with different path of understanding growth. Subjects couldn't fold back by themselves. They needed invocative intervention. It's suggested that students explore non-routine problem to develop their understanding. Moreover, since the result didn't show exploration in structuring and inventising layer, it's suggested that further research select proper problem to be explored by the students.

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